Sections 2.6, 3.8, 3.9, 3.10

2.6 Trace and Breit-Wigner formulas:

Recall: We are back in 1 dimension, $P_V = D_x^2 + V(x)$, and if A is a self-adjoint operator on a Hilbert space H and $\varphi \in L^\infty(\mathbb{R})$ then function calculus allows one to make sense of $\varphi(A)$.

Goals: 1) Relate the scattering matrix to the trace of $f(P_V) - f(P_0)$ (Birman-Krein formula)

2) Prove a version of the Breit-Wigner approximation to measure the effect of resonances on the spectrum.

3) Prove a Poisson formula to relate the trace of the wave group to a sum of resonances $\sum e^{-i\lambda_j|t|}$.

Later we will prove analogues in higher dimensions - Melrose trace formula

Theorem 2.19 (Birman-Krein formula in one dimension): Suppose that $V \in L^\infty_{comp}(\mathbb{R};\mathbb{R})$. Then for $f \in \mathscr{S}(\mathbb{R})$ the operator $f(P_V) - f(P_0)$ is of trace class and

$$\operatorname{tr}(f(P_V) - f(P_0)) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}(S(\lambda)^{-1}\partial_\lambda S(\lambda))\,d\lambda + \sum_{k=1}^{K} f(E_k) + \frac{1}{2}(m_R(0) - 1)f(0),$$

where $S(\lambda)$ is the scattering matrix and $E_K < \dots < E_1 < 0$ are the negative eigenvalues of P_V.

Interpretation: Recall that if one wants to solve $(P_V - \lambda^2)u = 0$ on (a,b) with Dirichlet boundary conditions one gets a discrete spectrum of sol'ns $(i\sqrt{-E_k}, u_k)$ (λ_j, u_j) with

$E_N < \dots < E_1 \leq 0 < \lambda_0^2 < \lambda_1^2 < \dots \to \infty$

For $f \in \mathcal{S}(\mathbb{R})$ this gives $\operatorname{tr} f(P_V^{\text{dirichlet}}) = \sum_{j=0}^{\infty} f(\lambda_j)^2 + \sum_{k=1}^{N} f(E_k)$

which one can interpret as $\operatorname{tr} f(P_V^{\text{dirichlet}}) = \int_0^\infty f(\lambda^2) \frac{dN(\lambda)}{d\lambda} d\lambda + \sum_{k=1}^{N} f(E_k)$

where $N(\lambda) = |\{\lambda_j^2 ; \lambda_j^2 \leq \lambda\}|$ counts the positive eigenvalues of $P_V^{\text{dirichlet}}$

This gives the following formal correspondences

Confined system (discrete spectrum)	Scattering (continuous spectrum)
$N(\lambda) \longleftrightarrow$	$\sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda)$
	i.e. $\sigma'(\lambda) = \frac{1}{2\pi i} \operatorname{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda))$

The extra term $\sigma(0) = \frac{1}{2}(m_R(0)+1)$ comes from jumping over a branch of the logarithm.

I will omit the proof as it is several pages long - we next focus on getting a formula for σ' in terms of resonances.

<u>Theorem 2.20 (Breit-Wigner approximation)</u>: Suppose that

$V \in L^\infty_{comp}(\mathbb{R};\mathbb{R})$. Then

$$(2.16) \quad \frac{1}{2\pi i} \operatorname{tr}(\partial_\lambda S(\lambda) S(\lambda)^{-1}) = -\frac{1}{\pi} |\operatorname{chsupp} V| - \frac{1}{\pi} \sum_j \frac{\operatorname{Im} \lambda_j}{|\lambda - \lambda_j|^2} \qquad \lambda \in \mathbb{R}$$

where the sum is over all non-zero resonances of P_V.

To go a little further we define the following distribution on $\mathbb{R}$.

$$\text{P.V.} \sum_j e^{-i|t|\lambda_j}(\varphi) := \lim_{R\to\infty} \sum_{|\lambda_j| \leq R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j |t|} dt \qquad \varphi \in C_c^\infty((-L,L)),$$ the sum is over the resonances.

To see that this is a well-defined distribution, write $e^{-i\lambda t} = \partial_t\left(\frac{i}{\lambda} e^{-i\lambda t}\right)$ and integrate by parts twice to get

$$P.V. \sum_{\lambda_j \neq 0} e^{-i|t|\lambda_j}(\varphi) = \lim_{R\to\infty} \sum_{0<|\lambda_j|\leq R} \int_0^\infty (\varphi(t)+\varphi(-t)) e^{-i\lambda_j t}\,dt.$$

$$= \lim_{R\to\infty} \sum_{0<|\lambda_j|\leq R} \left(\frac{2i}{\lambda_j}\varphi(0) + \mathcal{O}\left(\frac{\sup|\varphi''|}{|\lambda_j|^2}\right)\right)$$

One can now argue that the RHS converges using symmetry of the resonances under $\lambda_j \mapsto -\bar{\lambda}_j$ to write $\sum_{0<|\lambda_j|\leq R} \frac{2i}{\lambda_j} = \sum_{0<|\lambda_j|\leq R} \frac{2\,\mathrm{Im}\,\lambda_j}{|\lambda_j|^2}$

then using Carleman estimates to get $\sum_j \frac{|\mathrm{Im}\,\lambda_j|}{|\lambda_j|^2} < \infty$,

This allows us to write down the <u>trace formula for resonances</u>

<u>Thm 2.21 (Trace Formula for resonances)</u> Suppose that $V \in L^\infty_{comp}(\mathbb{R};\mathbb{R})$. Then for $\varphi \in C_c^\infty(\mathbb{R})$ the operator $\int_{\mathbb{R}} \varphi(t)(\cos t\sqrt{P_V} - \cos t\sqrt{P_0})\,dt$ is of trace class and

$$2\,\mathrm{tr}\,(\cos t\sqrt{P_V} - \cos t\sqrt{P_0}) = P.V. \sum_{\lambda\in\mathbb{C}} m_R(\lambda) e^{-i\lambda|t|} - 2|\mathrm{chsupp}\,V|\,\delta_0(t) - 1. \qquad (*)$$

in the sense of distributions on $\mathbb{R}$.

<u>Remark</u> If one was working on $[a,b]$ one would have

$$2\,\mathrm{tr}\cos t\sqrt{P_V^{\text{dirichlet}}} = \sum_{\lambda^2\in \mathrm{Spec}(P^{\text{dirichlet}})} e^{-i\lambda t}.$$

<u>Sketch of a proof of Thm 2.21</u>: We have $(2\cos t\sqrt{P_V})(\varphi) = f(P_V)$ where $f(z) = \hat{\varphi}(\sqrt{z}) + \hat{\varphi}(-\sqrt{z})$. Using the def'n of this P.V. distribution $(*)$ is equivalent to $\mathrm{tr}(f(P_V) - f(P_0)) = \lim_{R\to\infty} \sum_{|\lambda_j|\leq R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j|t|}dt - 2|\mathrm{chsupp}\,V|\varphi(0) - \hat{\varphi}(0)$

By the Birman-Krein Formula:

$$\mathrm{tr}\left(f(P_V)-f(P_0)\right)=\frac{1}{2}\int_{-\infty}^{\infty} f(\lambda^2)\sigma'(\lambda)\,d\lambda+\sum_{k=1}^{K} f(E_k)+\frac{1}{2}\left(m_R(0)-1\right)f(0)$$

where $\sigma(\lambda)=\ln\det S(\lambda)/2\pi i$.

This reduces the proof to showing that

$$\int_{\mathbb{R}}\hat{\varphi}(\lambda)\sigma'(\lambda)\,d\lambda+m_R(0)\,\hat{\varphi}(0)+\sum_{\mathrm{Im}\,\lambda_j>0}\left(\hat{\varphi}(\lambda_j)+\hat{\varphi}(-\lambda_j)\right)$$

$$=\lim_{R\to\infty}\sum_{|\lambda_j|\le R}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}\,dt-2|\mathrm{ch\,supp}\,V|\,\varphi(0).$$

Using that $t\mapsto e^{-i|t|\lambda_j}$ is a tempered distribution for $\mathrm{Im}\,\lambda_j<0$ and the symmetry $\lambda_j\mapsto-\overline{\lambda_j}$ we get

$$\mathcal{F}_{t\mapsto\lambda}\Big(\sum_{\substack{|\lambda_j|\le R\\ \mathrm{Im}\,\lambda_j<0}} e^{-i\lambda_j|t|}\Big)=\sum_{\substack{|\lambda_j|\le R\\ \mathrm{Im}\,\lambda_j<0}}\frac{2|\mathrm{Im}\,\lambda_j|}{|\lambda-\lambda_j|^2}$$

Hence

$$\lim_{R\to\infty}\sum_{\substack{|\lambda_j|\le R\\ \mathrm{Im}\,\lambda_j<0}}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}\,dt\overset{\text{Parseval}}{=}\frac{1}{\pi}\int_{\mathbb{R}}\hat{\varphi}(\lambda)\sum_{\mathrm{Im}\,\lambda_j<0}\frac{|\mathrm{Im}\,\lambda_j|}{|\lambda-\lambda_j|^2}\,d\lambda.$$

Plugging in the Breit-Wigner approximation and using $\int_{\mathbb{R}}\hat{\varphi}(\lambda)\,d\lambda=2\pi\varphi(0)$ it remains to prove that

$$\frac{1}{\pi}\int_{\mathbb{R}}\hat{\varphi}(\lambda)\sum_{\mathrm{Im}\,\lambda_j>0}\frac{\mathrm{Im}\,\lambda_j}{|\lambda-\lambda_j|^2}\,d\lambda+\sum_{\mathrm{Im}\,\lambda_j>0}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}\,dt=\sum_{\mathrm{Im}\,\lambda_j>0}\left(\hat{\varphi}(\lambda_j)+\hat{\varphi}(-\lambda_j)\right)$$

$$=2\sum_{\mathrm{Im}\,\lambda_j>0}\int_{\mathbb{R}}\varphi(t)\cos(\lambda_j t)\,dt.$$

To prove this one uses $e^{i\lambda_j|t|}+e^{-i\lambda_j|t|}=2\cos(\lambda_j t)$ and argues as we did for $\mathrm{Im}\,\lambda_j<0$.

Section 3.8 More on distorted Plane waves

Since distorted plane waves are used to define the scattering matrix, it makes sense to study them further. We start with an explicit form of Stone's Formula

Thm 3.47 (Stone's formula for P_V) Let $V\in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$. For $\lambda\in\mathbb{R}\setminus\{0\}$ and $\omega\in S^{n-1}$ define

$$(3.8.1)\quad e(\lambda,\omega,x):=e^{-i\lambda\langle x,\omega\rangle}-R_V(\lambda)(Ve^{-i\lambda\langle\cdot,\omega\rangle})(x).$$

Then $$(3.8.2)\quad \overline{e(\lambda,\omega,x)}=e(-\lambda,x,\omega)$$ and

$$(3.8.3)\quad R_V(\lambda,x,y)-R_V(-\lambda,x,y)=\frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}\int_{S^{n-1}}e(\lambda,\omega,x)\overline{e(\lambda,\omega,y)}\,d\omega$$

If one defines $$dE_\lambda(\lambda,x,y)=\int_{S^{n-1}}e(\lambda,\omega,x)\overline{e(\lambda,\omega,y)}\,d\omega\,\frac{\lambda^{n-1}d\lambda}{(2\pi)^n}$$

then

$$(3.8.4)\quad P_V=\sum_{k=1}^{K}E_k u_j\otimes\overline{u_j}+\int_0^\infty\lambda^2\,dE_\lambda,\quad I=\sum_{k=1}^{K}u_k\otimes\overline{u_k}+\int_0^\infty dE_\lambda.$$

where the u_k are normalized eigenfunctions of P_V corresponding to the eigenvalues E_k with $E_K\le\dots\le E_1\le 0$

Remarks (3.8.3) for $V\equiv 0$ was proved all the way back in Thm 3.4. The functions in (3.8.1) are the distorted plane waves; i.e. $e^{-i\lambda\langle x,\omega\rangle}$ is a free plane wave. We later discuss how the scattering matrix intertwines $e(-\lambda,-\omega,x)$ and $e(\lambda,\omega,x)$.

Sketch of a proof:

(3.8.2) Self-adjointness of P_V (V is real-valued) implies that for $\mathrm{Im}\,\lambda > 0$, $R_V(\lambda)^* = R_V(-\bar{\lambda})$. But since we know from Section 3.2 that $R_V(\lambda, x, y) = R_V(\lambda, y, x)$ we obtain:

$$\overline{R_V(\lambda)\bar{u}} = R_V(\lambda)^* u = R_V(-\bar{\lambda})u. \qquad \mathrm{Im}\,\lambda \geq 0$$

Hence for $\lambda \in \mathbb{R}$, $\overline{e(\lambda, \omega, x)} = e^{i\lambda\langle x,\omega\rangle} - \overline{R_V(\lambda)(Ve^{i\lambda\langle x,\omega\rangle})}$

$$= e^{i\lambda\langle x,\omega\rangle} - R_V(-\lambda)(Ve^{i\lambda\langle x,\omega\rangle}) = e(-\lambda, \omega, x).$$

(3.8.3) is a bit tricky, but implies (3.8.4) by Stone's thm in appendix B.

For (3.8.3) we first test against an arbitrary test function to get that (3.8.3) is equivalent to

$$\langle (R_V(\lambda) - R_V(-\lambda))\varphi, \varphi\rangle = \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}\int_{S^{n-1}}\left|\int_{\mathbb{R}} e(\lambda, \omega, x)\varphi(x)dx\right|^2 d\omega$$

holding $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$.

For $\mathrm{Im}\,\lambda < 0$ $R_V(-\lambda)^* = R_V(\bar{\lambda})$ so by continuation

$$\langle R_V(-\lambda)\varphi, \varphi\rangle = \langle \varphi, R_V(\bar{\lambda})\varphi\rangle \quad \lambda \in \mathbb{C}.$$

Using this and $(P_V - \lambda^2)R_V(\pm\lambda)\varphi = \varphi$ we get

suppose φ is supported in $B(0,R)$

$$\langle (R_V(\lambda) - R_V(-\lambda))\varphi, \varphi\rangle = \langle R_V(\lambda)\varphi, \varphi\rangle - \langle \varphi, R_V(\bar{\lambda})\varphi\rangle =$$

$$= \langle R_V(\lambda)\varphi, \varphi\rangle_{L^2(B_R)} - \langle \varphi, R_V(\bar{\lambda})\varphi\rangle_{L^2(B_R)}$$

$$= \langle R_V(\lambda)\varphi, (P_V - \lambda^2)R_V(\lambda)\varphi\rangle_{L^2(B_R)} - \langle (P_V - \lambda^2)R_V(\lambda)\varphi, R_V(\lambda)\varphi\rangle_{L^2(B_R)}$$

Cancellations $\langle \Delta R_V(\lambda)\varphi, R_V(\lambda)\varphi\rangle_{L^2(B_R)} - \langle R_V(\lambda)\varphi, \Delta R_V(\lambda)\varphi\rangle_{L^2(B_R)}$

Green's formula then yields that

$$\langle R_V(\lambda) - R_V(-\lambda)\varphi, \varphi\rangle = 2i\,\mathrm{Im}\int_{\partial B(0,R)} \partial_r [R_V(\lambda)\varphi](y)\, \overline{R_V(\lambda)\varphi(y)}\, dS(y).$$

The idea now is to prove the asymptotics

$$R_V(\lambda, r\omega, x) = \frac{e^{i\lambda r}}{r^{\frac{n-1}{2}}}\lambda^{\frac{n-3}{2}} c_n\, e(\lambda, y, \omega) + O(r^{-\frac{n+1}{2}}) \quad \text{as } r\to\infty$$

uniformly for y in compact sets, $c_n = \frac{1}{4\pi}\left(\frac{1}{2\pi i}\right)^{\frac{1}{2}(n-3)}$

inserting into and letting $R\to\infty$. $\square$

We now see how the scattering matrix intertwines distorted plane waves:

<u>Thm 3.49</u>: In the notation of Thm 3.47 define

$$(3.8.10)\quad E_V(\lambda)f(\omega) := \int_{\mathbb{R}^n} e(\lambda,\omega,x) f(x)\,dx \qquad f\in L^2_{comp}(\mathbb{R}^n)$$

$E_V(\lambda): L^2_{comp}(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$. Then $E_V(\lambda) = S(\lambda)\, J E_V(-\lambda)$ where $S(\lambda)$ is the scattering matrix & $Jf(\theta) = f(-\theta)$. In other words

$$(3.8.14)\qquad S(\lambda)e(-\lambda, -\bullet, x) = e(\lambda, \bullet, x).$$

The proof is not difficult using results from 3.7 & previous results we have discussed.

3.9 The Birman-Krein trace formula in higher dim.

We want to generalize the three thms from Section 2.6 to real valued potentials in odd dimensions. As it turns out, the Birman-Krein formula for $tr(f(P_V)-f(P_0))$ is valid in all dimensions and for quite general potentials; the trace formula fails in even dimensions and with general potentials.

We first need to know that $f(P_V)-f(P_0)$ is trace class.

<u>Thm 3.50 (Trace class property of $f(P_V)-f(P_0)$)</u> Suppose that $V\in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$. For $f\in \mathcal{S}(\mathbb{R})$ $f(P_V)-f(P_0)\in \mathcal{L}_1(L^2(\mathbb{R}^n))$ and $T_V: f\mapsto tr(f(P_V)-f(P_0))$ defines a tempered distribution. In addition, if $\mathbb{1}_{B(0,R)}$ is the indicator function of $B(0,R)$ then

$$\mathbb{1}_{B(0,R)}f(P_V)\in \mathcal{L}_1(L^2(\mathbb{R}^n)) \quad \text{and}$$

$$tr(f(P_V)-f(P_0)) = \lim_{R\to\infty} tr\, \mathbb{1}_{B(0,R)}(f(P_V)-f(P_0)).$$

<u>Thm 3.51 (The Birman-Krein formula)</u>: Suppose $V\in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$, $n\geq 3$ is odd. Then for $f\in\mathcal{S}(\mathbb{R})$

$$tr(f(P_V)-f(P_0)) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\, tr(S(\lambda)^{-1}\partial_\lambda S(\lambda))\,d\lambda + \sum_{k=1}^{K} f(E_k) + \frac{1}{2}\tilde{m}_R(0)f(0)$$

where $S(\lambda)$ is the scattering matrix & $E_K<\ldots<E_1\leq 0$ are the eigenvalues, $\tilde{m}_R(0)$ is non-zero only when $n=3$ (see section 3.3).

The proofs of 3.50/3.51 are reeeaaalllllyyy looonngg!

3.10 The Melrose trace formula

We now look to generalize Thm 2.21 to odd dimensions.

Arguing as before $\varphi \mapsto \sum_{\lambda \in \mathbb{C}} m_R(\lambda) \int_{\mathbb{R}} t^n \varphi(t) e^{-i\lambda|t|} dt$ is a distribution.

We now integrate by parts $n+1$ (instead of 2) times to see this.

This allows us to state the trace formula:

Thm 3.53 (Trace formula for resonances) Suppose that $V \in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$ $n \geq 3$ odd. Then (3.10.2) $2t^n tr(\cos t\sqrt{P_V} - \cos t\sqrt{P_0}) = t^n \sum_{\lambda \in \mathbb{C}} m_+(\lambda) e^{-i\lambda|t|}$

where (3.10.3) $m_+(\lambda) := \begin{cases} m_R(\lambda) & \lambda \neq 0 \\ 2m_R(0) - \tilde{m}_R(0) & \lambda = 0. \end{cases}$

We can als nicely factor the scattering matrix

Thm 3.54 (Factorization of the scattering matrix) Suppose that $V \in L^\infty(\mathbb{R}^n;\mathbb{C})$, $n \geq 1$ is odd. Then $\det S(\lambda) = (-1)^m e^{g(\lambda)} \frac{P(-\lambda)}{P(\lambda)}$

(3.10.4) $P(\lambda) := \prod_{\mu \neq 0} E_n(\lambda/\mu)^{m_R(\mu)}$ $\quad E_n(z) := (1-z)e^{z + z^2/2 + \dots + z^n/n}$.

$g(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda$.

If $V \in L^\infty_{comp}(\mathbb{R}^n;\mathbb{R})$ then $P(-\lambda) = \overline{P(\bar{\lambda})}$, $a_n \in i\mathbb{R}$, $m = \tilde{m}_R(0)$

$\square$

The proofs are rather involved thus omitted.